Jonas Kremer WS 2017/2018

Probability theory

Exercise Sheet 7

Exercise 1 *(4 Points)*
Let $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R})$. Examine if there exists a measure $\mu \in \mathcal{P}(\mathbb{R})$ such that μ_n converges weakly to μ as $n \to \infty$, where

(a) μ_n is the binomial distribution with parameters n and λ/n, $\lambda > 0$, i.e.

$$\mu_n(A) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \delta_k(A), \quad A \in \mathcal{B}(\mathbb{R}).$$

(b) μ_n is the the uniform distribution on $(-n, n)$, i.e.

$$\mu_n(\mathrm{d}x) = \mathbb{1}_{(-n,n)} \frac{1}{2n} m(\mathrm{d}x).$$

Exercise 2 *(4 Points)*
Suppose that $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{P}(E)$, $\mu \in \mathcal{P}(E)$ admit density functions p_n, p with respect to a (non necessarily finite) measure λ on E. ($\mathrm{d}\mu_n = p_n \mathrm{d}\lambda$ and $\mathrm{d}\mu = p\mathrm{d}\lambda$.)

(a) Suppose $p_n \to p$ λ-almost surely as $n \to \infty$. Conclude that μ_n converges weakly to μ as $n \to \infty$.

(b) Find a counterexample showing that the converse in (b) is not true in general.

Exercise 3 *(6 Points)*
Let μ be a probability measure on $\mathbb{R}$ and let $F(t) := \mu((-\infty, t])$.

(a) Prove that $\lim_{t\to-\infty} F(t) = 0$, $\lim_{t\to\infty} F(t) = 1$, F is monotonically increasing and that F is right-continuous.

(b) Prove that F has at most countably many point of discontinuity.

What does *'F is continuous at t'* means in terms of μ? Give an example for μ where F is continuous and an example where F is only discontinuos at $t = 0$.

Exercise 4 *(4 Points)*
Prepare a talk on Portmanteau's Theorem and the idea of its proof.